Sandpiles and diffusion-limited reactions

Joachim Krug*
IBM Research Division, T. J. Watson Research Center, Yorktown Heights, New York 10598
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We establish the equivalence between a one-dimensional sandpile model recently introduced by Chhabra *et al.* [Phys. Rev. A (to be published)], and the diffusion-limited reaction $A+A\rightarrow 0$ with a point source. The equivalence is used to predict the power-law decay of the avalanche size distribution, $P(s)\sim s^{-3}$, in agreement with numerical simulations. This result differs from the mean-field treatment of Chhabra *et al.*, demonstrating the importance of fluctuations in this low-dimensional system.

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Sandpile models of self-organized criticality [1] are deterministic cellular automata which are driven to a statistically steady state by the repeated application of local stochastic perturbations. Each perturbation consists of adding a small amount (a "grain") of a conserved quantity ("sand") to the system. Whenever a local stability criterion is violated, sand is redistributed locally according to the automaton rules. The amount of sand (or the number of sites) affected by a single perturbation constitutes the size of an avalanche. Since sand can leave the system only at the open boundaries, a long-ranged transport flow has to be maintained in the steady state. In many cases this is sufficient [2–4] to enforce the establishment of criticality in the sense of a broad, scaling size distribution of avalanches.

A particularly simple, one-dimensional sandpile, the local limited (L^2) model, was introduced by Kadanoff and co-workers [2]. In this model an integer height variable H_i is associated with each site $i=1,\ldots,L$ of a one-dimensional lattice. A configuration is stable if all slopes satisfy $S_i=H_i-H_{i-1}\leq 2$. The system is perturbed by adding a grain ($H_i\rightarrow H_i+1$) at a randomly chosen site. If, as a consequence, the stability criterion is violated, two grains of sand are moved downhill from site i to site $i-1$. This procedure is repeated until all sites are again stable, at which point another grain is added. The system is open at $i=1$ and closed at $i=L$, i.e., the stationary sandpile slopes to the left.

Carlson and co-workers [5] pointed out that sites with $S_i\leq 0$ ("troughs" [5] or "traps" [7]) play a crucial role in the dynamics of the L^2 model. When an avalanche is initiated by adding a grain to a site j with $S_j=2$, all the sites between j and the position j_R of the nearest trap in the uphill (right) direction (or, if such a trap does not exist, the wall at $i=L$) also become unstable, and a total of $2(j_R-j)$ grains (two per site) is set into motion. The avalanche then slides downhill until its front end encounters the first trap in the downhill (left) direction, where it stops. If there is no trap between j and the abyss at $i=1$ the avalanche falls off the pile. Several diverging length scales can be associated with the spatial distribution of trap sites [3,5,8] and the resulting avalanche distribution is surprisingly complex [2,7]. In an attempt to make the problem more tractable Chhabra, Feigenbaum, Kadanoff, Kolan, and Procaccia [7] (CFKKP) recently introduced a truncated version of the L^2 model, the L^3 model, in which traps are not allowed to form. Consequently the slopes can only take on two values, $S_i=1,2$ [9]. This restriction is enforced by starting with an allowed configuration and simply suppressing all perturbations which would lead, after the termination of all ensuing avalanches [10], to a stable configuration where some slope $S_i\leq 0$. Given the importance of trap sites [5], this modification is expected to lead to a behavior which is qualitatively different from the original model. CFKKP solved the L^3 model in a mean-field approximation and found a pure power-law avalanche size distribution $P(s)\sim s^{-4}$, where s denotes the number of grains leaving the system during an avalanche. Here I show that the L^3 model is in fact equivalent to the much studied diffusion-limited annihilation reaction $A+A\rightarrow 0$ [11–13], with the slope playing the role of the diffusing species and the avalanches corresponding to the annihilation events. This equivalence is then used to derive the *exact* form of the avalanche distribution.

The main simplifying feature of the L^3 model is the *locality* of the slope dynamics, i.e., dropping a grain on a site i only affects [10] the slopes at i and $i+1$. Following CFKKP we introduce the variables $\epsilon_i=S_i-1=0,1$. In order to maintain the restriction $\epsilon_i\geq 0$ no sand can be added to sites i with $(\epsilon_i,\epsilon_{i+1})=(0,0)$, since such a move would create a trap at $i+1$. Below we list the outcomes of the remaining three allowed moves [7]:

$$(\epsilon_i,\epsilon_{i+1})=(0,1)\rightarrow(1,0)\ , \tag{1}$$

$$(\epsilon_i,\epsilon_{i+1})=(1,0)\rightarrow(0,1)\ , \tag{2}$$

$$(\epsilon_i,\epsilon_{i+1})=(1,1)\rightarrow(0,0)\ . \tag{3}$$

Move (1) corresponds simply to the addition of a grain to a stable site. In move (2) site i becomes unstable, but at the same time a trap is created at $i+1$ which limits the ensuing avalanche to a single site (two grains). Since there are no traps in the downhill direction, the avalanche falls off the pile. In the process the trap at $i+1$ is filled [5], so that the resulting configuration is again trap-free [10]. Finally, in move (3) site i is destabilized without the creation of a trap at a neighboring site.

Consequently the instability spreads all the way to the wall, and an avalanche of length $L-i+1$ [mass $2(L-i+1)$] slides off the pile. If a grain is added at site $i=L$, ϵ_L changes according to $\epsilon_L \rightarrow 1-\epsilon_L$, corresponding to the boundary condition [7] $\epsilon_{L+1}=1$.

In order to establish the correspondence with diffusion-limited annihilation we now interpret $\epsilon_i=1$ (0) to denote the presence (absence) of an A particle at site i. It is then evident that moves (1) and (2) correspond to the diffusive motion of particles, while move (3) describes the pairwise annihilation of two particles at neighboring sites. Note that the reaction rate is not unity: A pair of particles will react only if the *leftmost* one (at site i) is picked for the next move. However, the value of the reaction rate is irrelevant provided it is nonzero [11]. The avalanche distribution consists of two parts, small avalanches (of unit length) arising from the diffusion step (2) and large avalanches (of all lengths up to the system size) associated with the reaction step (3). Since small avalanches only involve a single particle, their probability is proportional to the particle density, while the probability for a large avalanche of length r (mass $s=2r$) is equal to the reaction rate at the distance $r=L+1-i$ from the wall.

The boundary condition $\epsilon_{L+1}=1$ provides a steady input of particles at the closed end of the sandpile. We thus expect the stationary particle density to decay as a function of the distance r from the boundary. In a continuum approximation, neglecting density fluctuations, the density profile can be computed from the equation [12]

$$\frac{\partial \epsilon(r,t)}{\partial t} = D\frac{\partial^2 \epsilon}{\partial r^2} + \sigma\delta(r) - k\epsilon^2 , \tag{4}$$

where D is the diffusion constant, σ denotes the input rate, and k the reaction rate. The stationary solution of (4) is readily seen to decay as $1/r^2$, in agreement with the detailed mean-field calculation of CFKKP [7]. Since, within this approximation, the reaction rate is proportional to ϵ^2, we also obtain the avalanche distribution $P(s) \sim s^{-4}$ as found by CFKKP.

However, diffusion-limited reactions are known to behave anomalously in low dimensions [11]. For the $A+A \rightarrow 0$ reaction, the correct behavior both in the transient and in the steady-state regime can be obtained by replacing the quadratic term in (4) by an effective reaction rate [13] proportional to ϵ^X, where $X=1+2/\tilde{d}>2$ if the spectral dimension [14] $\tilde{d}$ of the medium is less than 2. For a regular, one-dimensional lattice $\tilde{d}=d=1$, so $x=3$ and (4) predicts a $1/r$ decay of the density profile, in agreement with an exact calculation for a related one-dimensional lattice model [12]. Since the size distribution of large avalanches is determined by the spatial decay of the reaction rate, we conclude that $P(s) \sim \epsilon^3 \sim s^{-3}$, in contrast to the s^{-4} decay obtained by CFKKP. As usual, the breakdown of the mean-field approximation implies the presence of strong correlations. Indeed, since microscopically the reaction rate at site i is given by $\langle \epsilon_i \epsilon_{i+1} \rangle$, we see that

$$\langle \epsilon_i \epsilon_{i+1} \rangle \sim 1/r^3 \ll \langle \epsilon_i \rangle \langle \epsilon_{i+1} \rangle \sim 1/r^2 , \tag{5}$$

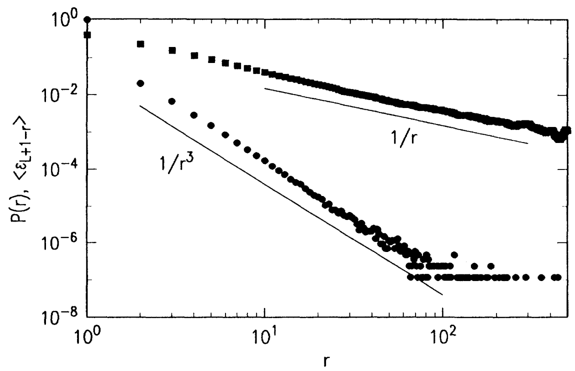

FIG. 1. Simulation results for an L^3 sandpile of size $L=500$. The squares show the reduced slope $\langle \epsilon_i \rangle$ as a function of the distance $r=L+1-i$ from the closed boundary, and the dots show the probability distribution for the avalanche length (the avalanche mass is $2r$). The full lines indicate the predicted power laws. The data were obtained by averaging over 2.5×10^6 attempted additions per site, which resulted in 8.54×10^6 avalanches.

i.e., particles at neighboring sites are strongly anticorrelated, despite the fact that the *density* of particles vanishes for $r \rightarrow \infty$.

Numerical simulations of the L^3 model were carried out in order to test these predictions (Fig. 1). Power-law fits to the density profile and the avalanche distribution show these quantities to decay as $r^{-0.96}$ and $r^{-3.16}$, respectively, in excellent agreement with the expected behavior.

In closing, it should be noted that the power-law decay of the avalanche distribution in the L^3 model does *not* arise as a consequence of mass balance requirements [2,3,4]. The $1/r$ decay of the density profile implies that the number of particles in a system of size L is of the order $\ln L$. Thus the probability that an attempt to add a grain of sand at a randomly chosen position on the pile will be successful is approximately $\ln L/L$, and the probability that such a move will trigger a small avalanche is of the same order. In contrast, the probability for a large avalanche is only of the order $1/L$. Since the $1/s^3$ size distribution of large avalanches has a finite first moment, we see that the fraction of the total mass flow through the system which is carried by large rather than small avalanches vanishes as $1/\ln L$ for large L, i.e., the large avalanches are irrelevant for maintaining mass balance. The prominence of the small avalanche peak at $r=1$ in Fig. 1 is an indication of this fact. As in a recently introduced noncritical variant of the L^2 model [6], the mass transport through small avalanches is sufficient to maintain stationarity because every avalanche that is triggered in the bulk of the system propagates all the way to the open boundary.

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*Present address: Institut für Festkörperforschung, Forschungszentrum Jülich, P.O. Box 1913, W-5170 Jülich, Germany.

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